OPTIMIZATION UNDER UNCERTAINTY:

State-of-the-Art and Opportunities

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A LONG RECOGNIZED NEED

“Those of us who were doing the planning right from the very beginning understood that the real problem was to be able to do planning under uncertainty.”

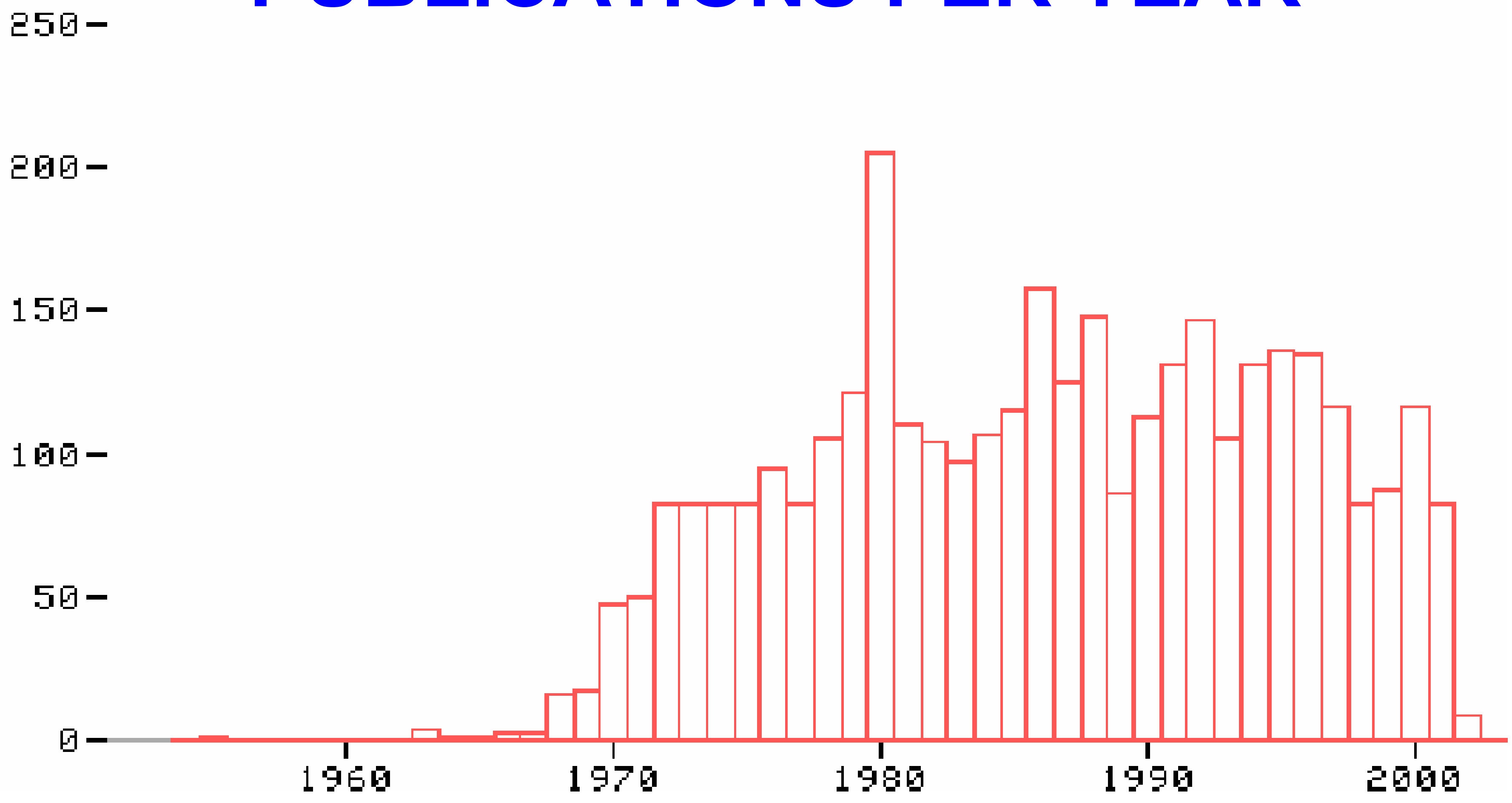
G. B. Dantzig, E-Optimization (2001)

Interviewed by Irv Lustig

THE FIRST PAPERS

- **Stochastic Programming**
  - Based on probability distributions for uncertain parameters
  - Minimize expected costs
    - » Beale (1955)
    - » Dantzig (1955)
    - » Tintner (1955)
  - Maximize system's ability to meet constraints
    - » Charnes & Cooper's chance-constraint programming (1959)
- **Fuzzy Programming**
  - Optimization over soft constraints
  - Bellman & Zadeh (1970)

STOCHASTIC PROGRAMMING PUBLICATIONS PER YEAR



- Maarten H. van der Vlerk. *Stochastic Programming Bibliography*. http://mally.eco.rug.nl/biblio/stoprog.html, 1996-2002
- Over 3500 papers on stochastic programming
  - 100 papers per year for the past 30 years

STILL A NEED

"Planning under uncertainty. This, I feel, is the real field we should all be working on."

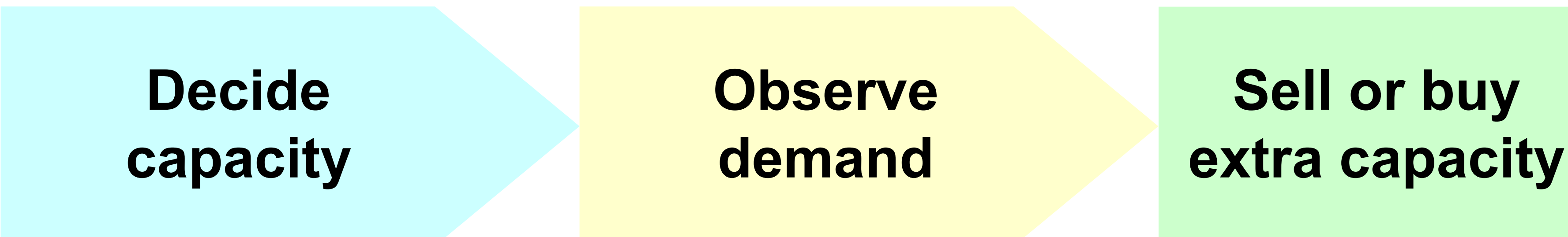
G. B. Dantzig, E-Optimization (2001)

PRESENTATION GOALS

- **Illustrate algorithmic challenges**
  - Stochastic programming
    - » Expectation minimization
    - » Chance-constrained
    - » Linear, integer, and nonlinear programming
  - Fuzzy programming
- **Review progress to date**
  - Computational state-of-the-art
- **Highlight contributions of the PSE community to optimization under uncertainty**

STOCHASTIC PROGRAMS

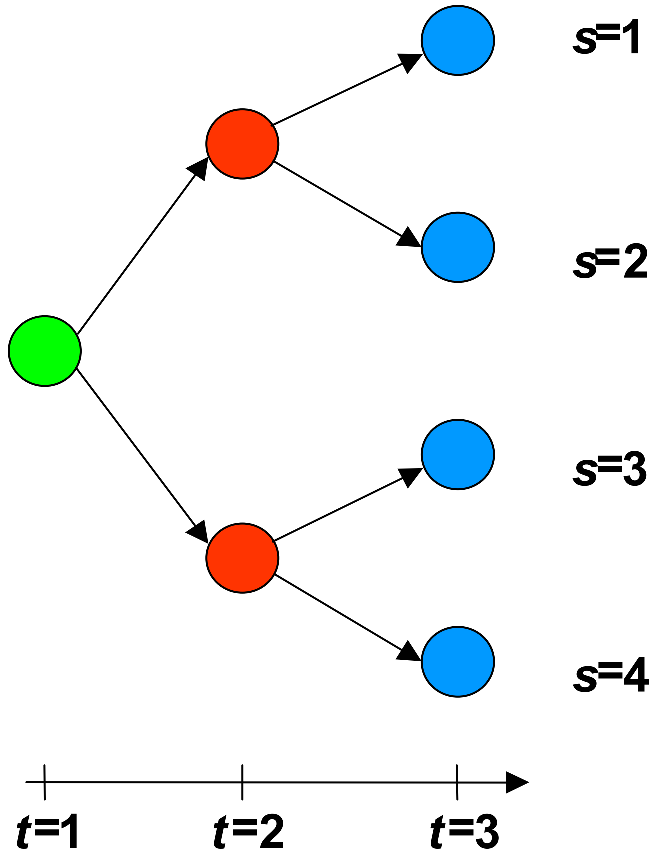
- **Multi-stage optimization problems with parameter uncertainties**
  - **Decisions do not affect the uncertainties**
  - **Finite number of decision stages**



- **Objective: Minimize expected total cost**

MODELING UNCERTAINTY

- **Assume: A finite sample space**
- **Uncertainty is modeled as a scenario tree**
- **A scenario is a path from the root to a leaf**



TWO-STAGE STOCHASTIC LP WITH RECOURSE

- **Decide $x \Rightarrow$ Observe scenario $\Rightarrow$ Decide y**
  - x **is the vector of first-stage variables**
  - y **is the vector of second-stage variables**
- **Objective: E[total cost]**
- **Second stage problem depends on first-stage decision and scenario realized**

$$\begin{aligned} \min \quad & cx + \sum_{s=1}^{S} p^s Q^s(x) \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0, \end{aligned}$$

where $Q^s(x) = \min\{f^s y \mid D^s y \geq h^s + T^s x\}.$

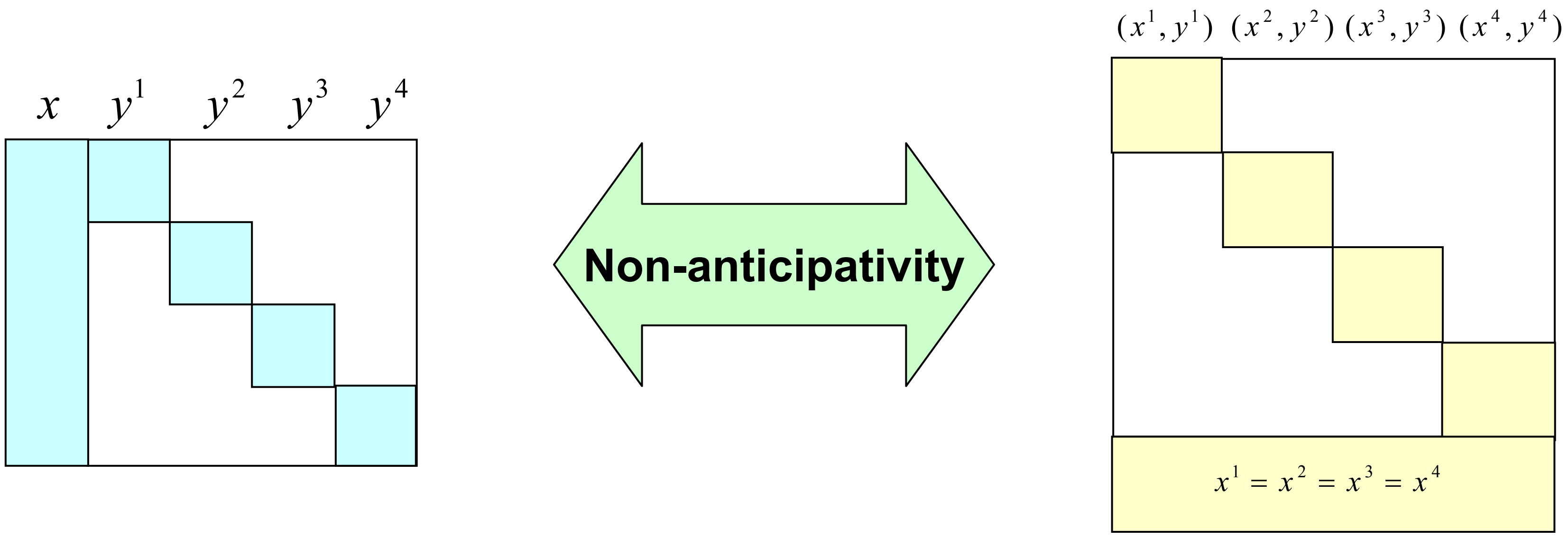
THE CHALLENGE

- **Consider 100 uncertain parameters**
- **Each parameter can take 3 values**
- **Total number of possible scenarios is $3^{100} = 5 \times 10^{47}$**

- **Explicit evaluation of the second-stage cost function is out of the question**

STOCHASTIC LP

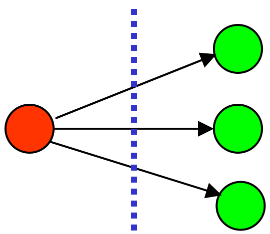
- $Q^s(x)$ **is the value function of a linear program**
  - **Piece-wise linear and convex**
  - **Convex programming methods are applicable**
- **Properties and algorithms extend to:**
  - **Multi-stage stochastic LP**
  - **First-stage integer variables**
- **Large scale LP with special structure**



DECOMPOSITION

Primal Methods	Dual Methods

Primal Methods

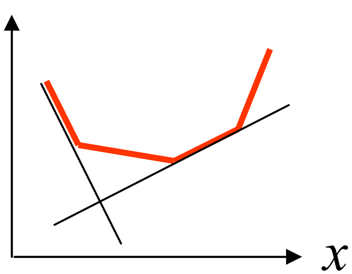


$$\min_{x \in X} cx + \sum_{s=1}^{S} p^s \theta^s$$

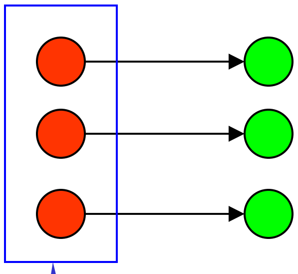
$$\text{s.t.} \quad \theta^s \geq a_1^s + b_1^s x \quad \forall s$$

$$\theta^s \geq a_2^s + b_2^s x \quad \forall s$$

$$\vdots$$



Dual Methods



Non-anticipativity

$$\sum_{s=1}^{S} H^s x^s = 0$$

$$\max_{\lambda} L(\lambda)$$

$$L(\lambda) = \sum_{s=1}^{S} \min_{\substack{x^s \in X \\ Dy^s \geq h^s + T^s x^s}} \{(c + \lambda H^s) x^s + p^s f^s y^s\}$$

SAMPLING APPROXIMATIONS

- "Interior" sampling methods
  - In each decomposition iteration, sample a few only scenarios
  - Dantzig and Infanger (1992), Infanger (1994)
- "Exterior" sampling methods
  - First sample a few scenarios, then solve stochastic LP with sampled scenarios only
  - Shapiro (1996)
- Desirable statistical convergence properties

STATE-OF-THE-ART IN COMPUTATIONS

- **Exact algorithms**
  - Birge (1997)
  - Millions of variables in deterministic equivalent
    - » 1000 variables
    - » 10 uncertain parameters, each with 3 possible values
  - Parallel computers
- **Sampling-based methods**
  - Linderoth, Shapiro and Wright (2002)
  - Computational grid
  - Up to 10^{81} scenarios
  - Within an estimated 1% of optimality

TWO-STAGE STOCHASTIC INTEGER PROGRAMMING

- **Second stage optimization problem involves combinatorial decisions**

- **Examples:**
  - **Resource acquisition (Dempster et al., 1983):**
    Acquire machines $\Rightarrow$ Observe processing times $\Rightarrow$ Schedule jobs
  - **Location-Routing (Laporte et al., 1989):**
    Locate depots $\Rightarrow$ Observe demand $\Rightarrow$ Route vehicles
  - **Crew recovery:**
    Assign crews $\Rightarrow$ Observe breakdown $\Rightarrow$ Recover crews

- $Q^s(x)$ **is the value function of an integer program**

THE CHALLENGE

$$\min \quad f(x_1, x_2) = -1.5x_1 - 4x_2 + \sum_{s=1}^{4} \frac{1}{4} Q^s(x_1, x_2)$$

$$s.t. \quad 0 \le x_1, x_2 \le 5$$

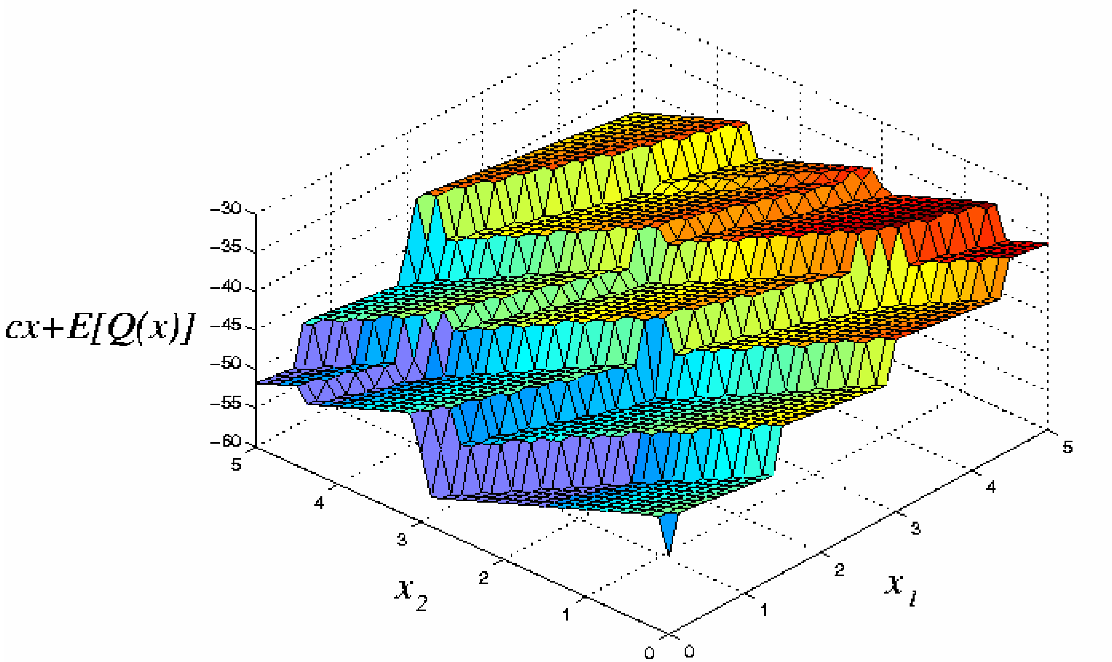
$$Q^s(x_1, x_2) := \min \quad -16y_1 - 19y_2 - 23y_3 - 28y_4$$

$$s.t. \quad 2y_1 + 3y_2 + 4y_3 + 5y_4 \le \omega_1^s - \frac{1}{2}x_1 - \frac{2}{3}x_2$$

$$6y_1 + y_2 + 3y_3 + 2y_4 \le \omega_2^s - \frac{2}{3}x_1 - \frac{1}{3}x_2$$

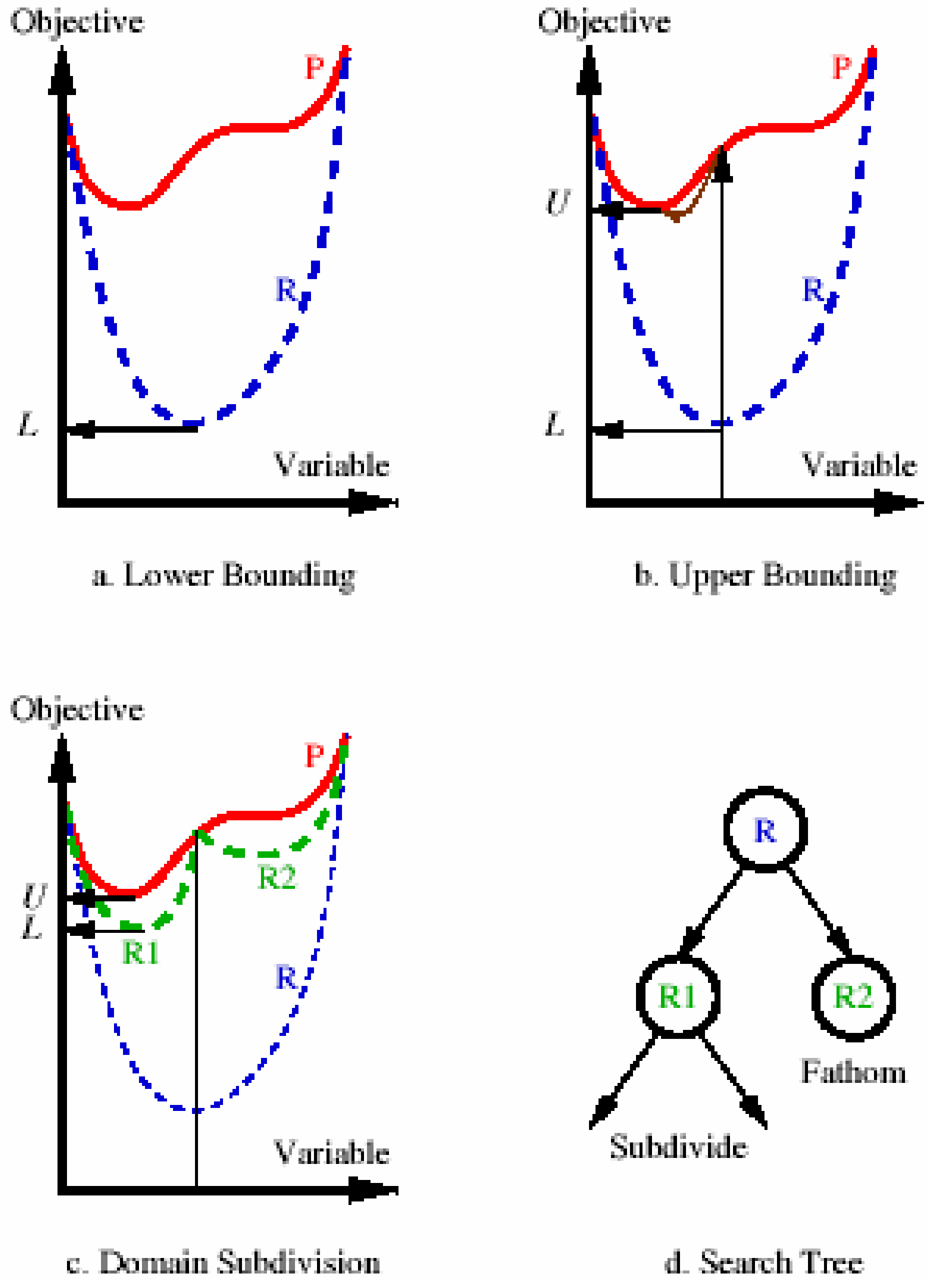
$$y_i \in \{0,1\} \quad \text{for } i = 1, \ldots, 4$$

where $(\omega_1, \omega_2) \in \{5,15\} \times \{5,15\}$



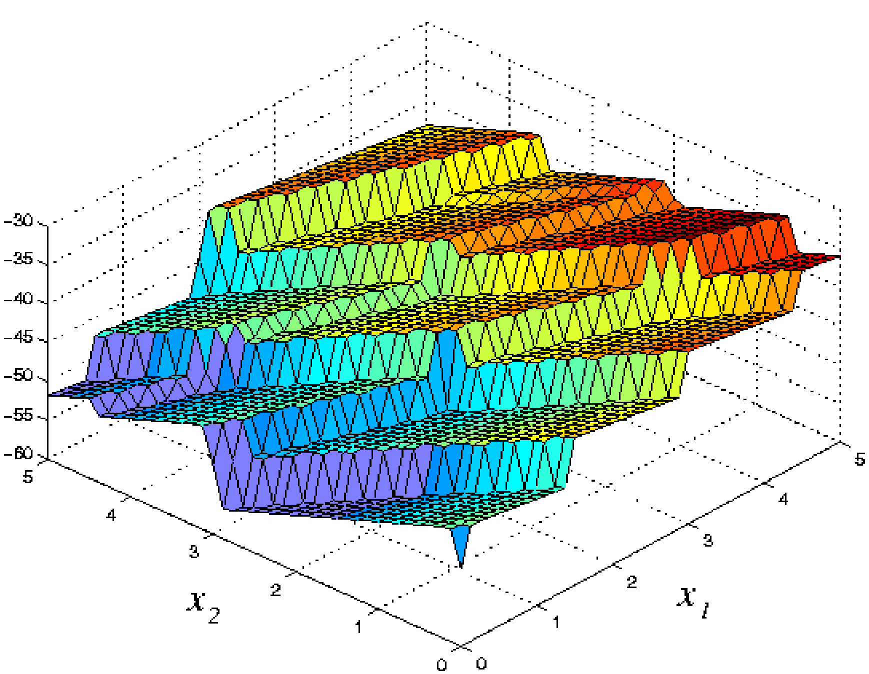
- **Discontinuous**
- **Highly non-convex**
- **Many local minima**

BRANCH-AND-BOUND



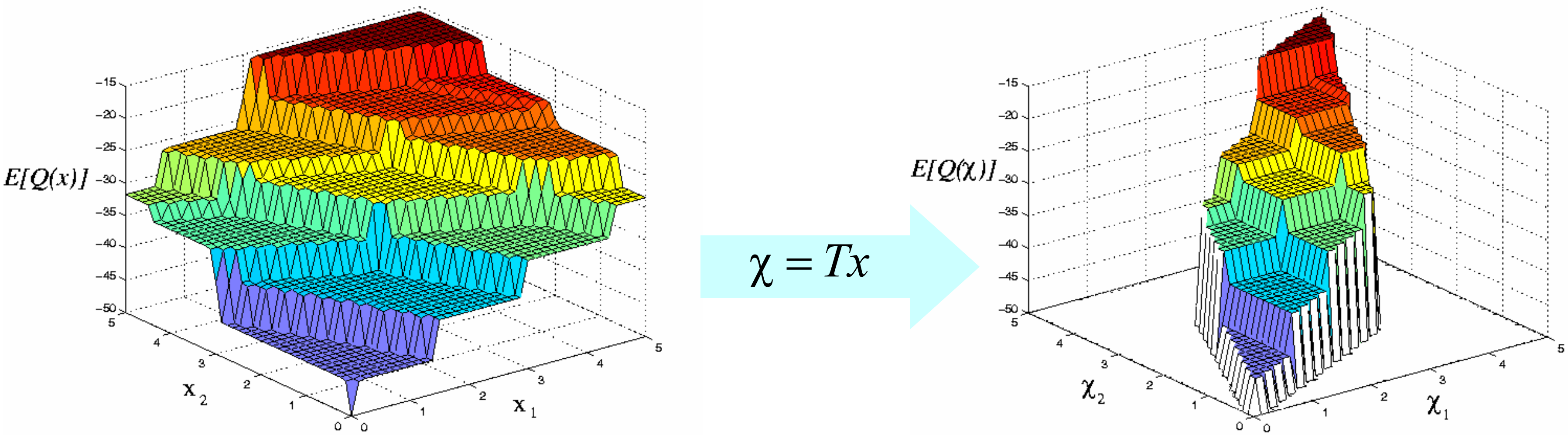
FINITENESS ISSUE

- With continuous first-stage variables, existing B&B algorithms are not finite



- The most common branching scheme is rectangular partitioning—branching along a variable axis
- The polyhedral shaped discontinuous pieces cannot be isolated by a finite number of rectangular partitions
- Requires infinite partitioning for lower and upper bounds to become equal

VARIABLE TRANSFORMATION



Ahmed, Tawarmalani and Sahinidis (Math Progr, 2003)

- Solve the problem in the space of the "tender variables"
- Variable transformation aligns discontinuities orthogonal to variable axes
- Discontinuities identified based on Blair and Jeroslow (1977) results
- Finite termination

IMPLEMENTATION

- **BARON is used to maintain the branch and bound tree**
  - **Constraint propagation & duality-based range reduction**
    - » **Ryoo and Sahinidis, 1995, 1996**
    - » **Shectman and Sahinidis, 1998**
    - » **Tawarmalani and Sahinidis, 2002**
  - **Tawarmalani and Sahinidis, *Convexification and Global Optimization in Continuous and Mixed-Integer Nonlinear Programming*, Kluwer Academic Publishers, Nov. 2002.**
- **Open partitions account for discontinuities**
- **OSL is the IP solver**
- **CPLEX is the LP solver**

COMPUTATIONAL RESULTS

TEST PROBLEMS

Problem	Binary Variables	Continuous Variables	Constraints
SIZES3	40	260	142
SIZES5	60	390	186
SIZES10	110	715	341

	JORJANI ('95) CPLEX B&B				CAROE ('98) B&B with Lagrangian Rel.				BARON			
Problem	LB	UB	nodes	CPU ¶	LB	UB	nodes	CPU ¶	LB	UB	nodes	CPU *
SIZES3	218.2	224.7	20000	1859.8	224.3	224.5	-	1000	224.4	224.4	260	70.7
SIZES5	220.1	225.6	20000	4195.2	224.3	224.6	-	1000	224.5	224.5	13562	7829.1
SIZES10	218.2	226.9	250000	7715.5	224.3	224.7	-	1000	224.2	224.7	23750	10000.0

¶ Digital Alpha 500 Mhz

* IBM RS/6000 133 MHz

ROBUSTNESS ISSUES

- Recourse model provides first-stage solution that optimizes expected second-stage cost
- This solution may be very bad under certain conditions

- Robust solutions: remain near-optimal irrespective of uncertain outcome

- Mulvey, Vanderbei and Zenios (1995)
  - May not lead to optimal second-stage decisions
  - King et al. (1997), Sen and Higle (1999)
  - Takriti and Ahmed (2002)

- More recent approaches
  - Ben-Tal and Nemirovski (2000)
  - Bertsimas (2002)

PROBABILISTIC PROGRAMMING

- **Also known as chance-constrained programming**
- **Focuses on reliability of the system**
- **LP with chance constraints:**

$$\begin{aligned} \min \quad & c^t x \\ \text{s.t.} \quad & P(Ax \geq b) \geq p \\ & x \geq 0 \end{aligned}$$

- **Consider**
  - **One constraint**
  - **Deterministic *A*=*a*^{t}**
  - **Uncertain *b*, with *F(β)=p***

$$\Rightarrow \quad \begin{aligned} \min \quad & c^t x \\ \text{s.t.} \quad & a^t x \geq \beta \\ & x \geq 0 \end{aligned}$$

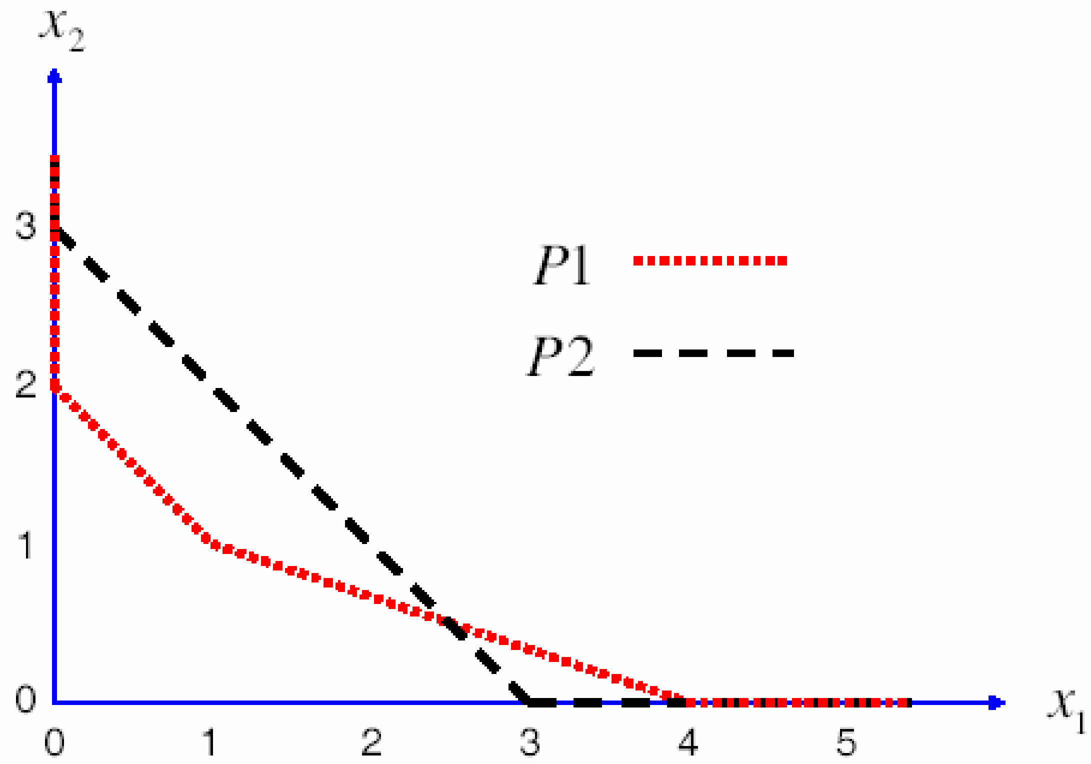
THE CHALLENGE

Consider

$$\min \quad c^t x$$

$$\text{s.t.} \quad P\begin{pmatrix} x_1 + x_2 \geq b_1 \\ x_1 + 3x_2 \geq b_2 \end{pmatrix} \geq 0.5$$

$$x_1 \geq 0, x_2 \geq 0$$



with

$$P(b_1 = 2, b_2 = 4) = 0.5$$

$$P(b_1 = 3, b_2 = 0) = 0.5$$

Probabilistic programming is a global optimization problem

FUZZY PROGRAMMING

- Considers uncertain parameters as fuzzy numbers
- Treats constraints as fuzzy sets
- Some constraint violation is allowed
- Bellman and Zadeh (1970)
  - Minimize largest constraint violation
- Flexible programming
  - Right-hand-side and objective uncertainty
- Possibilistic programming
  - Constraint coefficient uncertainty
  - Nonconvex optimization problem
    - Liu and Sahinidis (1997)
- Zimmermann (1991)
- Comparisons needed between SP and FP!

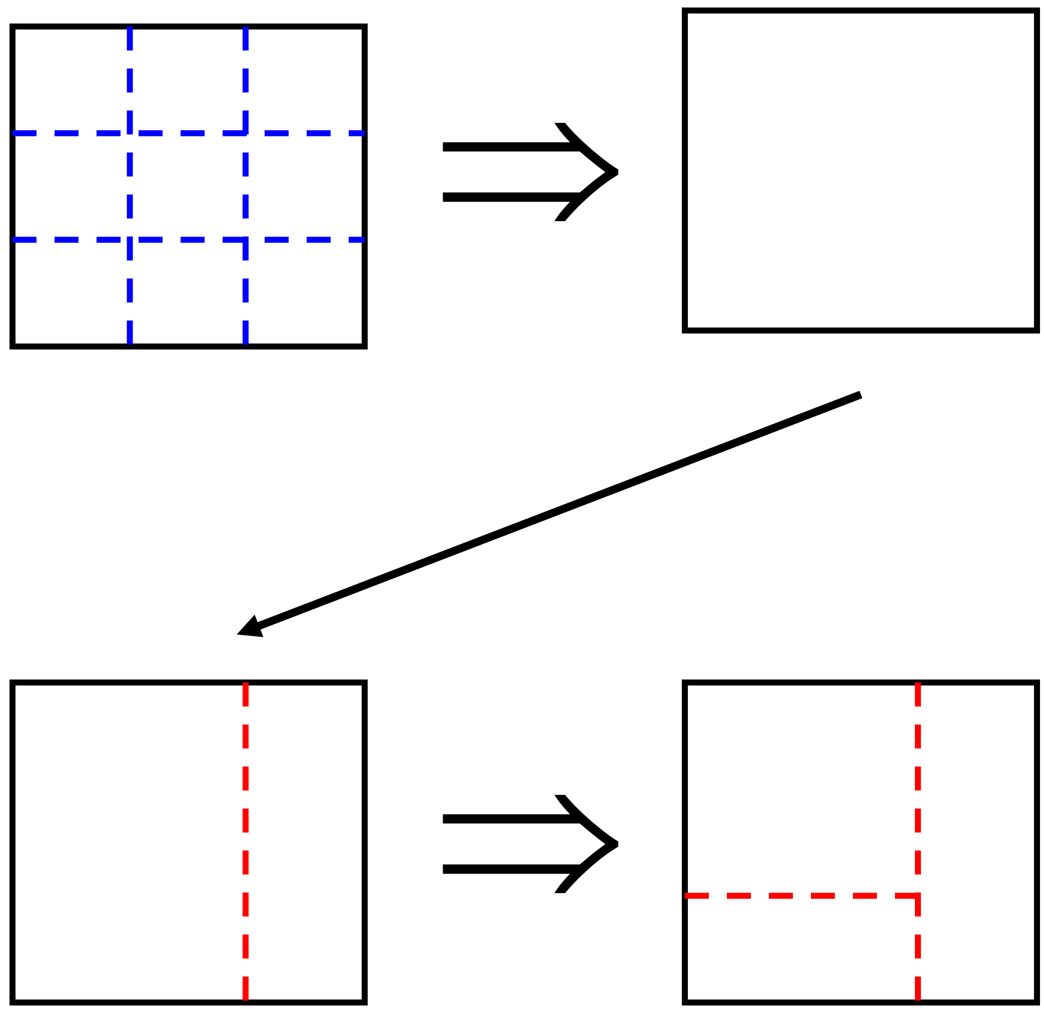
PSE DEVELOPMENTS

- **Aggregation-disaggregation for two-stage stochastic linear programs**
  - Clay and Grossmann (CACE, 1997)
- **Multiparametric programming for mixed-integer nonlinear programs**
  - Pistikopoulos et al. (IECR, 1999; CACE, 2002)
- **Finite algorithm for two-stage stochastic integer programs**
  - Ahmed, Tawarlamani and Sahinidis (Math Progr, 2003)
- **Approximation scheme for multistage stochastic integer programs for supply chain planning**
  - Ahmed and Sahinidis (Oper Res, 2003)

AGGREGATION-DISAGGREGATION FOR TWO-STAGE STOCHASTIC LPs

Clay and Grossmann (1997)

- Aggregation of probability space
- Rigorous lower and upper bounds
- Bounds sharpen by disaggregation



FEW SCENARIOS SUFFICE

Problem	Deterministic LP	Largest aggregate	Time (s)*
EX11E	6670 x 6940	430 x 460	34
EX11C	33710 x 35020	950 x 1000	179
EX11F	107,000 x 111,000	1470 x 1540	474
EX11G	260,000 x 270,000	2354 x 2458	1632
EX11H	539,000 x 560,000	2692 x 2809	2093
EX11I	1,704,000 x 1,770,000	5422 x 5644	16,286
EX11J	4,160,000 x 4,320,000	7450 x 7450	49,756

*: alpha workstation with 64 MB RAM

From: Clay and Grossmann (1997)

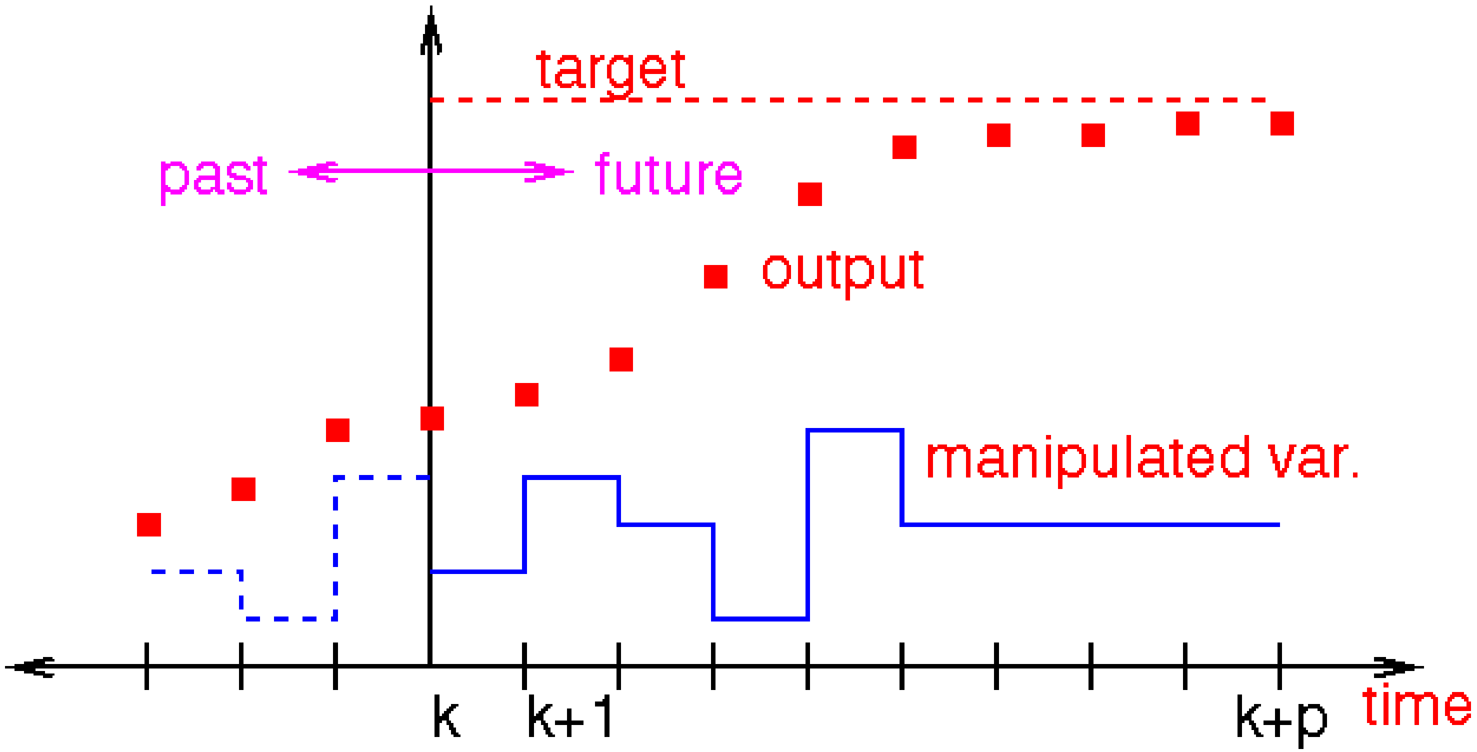
MULTI-PARAMETRIC MIXED-INTEGER NONLINEAR PROGRAMMING

- **Dua, Bozinis and Pistikopoulos (2002)**

$$
\begin{aligned}
z(\theta) = \quad & \min_{x} \quad \mathrm{c}x + \frac{1}{2}x^T Q x \\
& \text{s.t.} \quad Ax \leq b + F\theta \\
& \qquad\quad x \in \Re^n \\
& \qquad\quad \theta \in \Theta \subseteq \Re^{\mathrm{m}}
\end{aligned}
$$

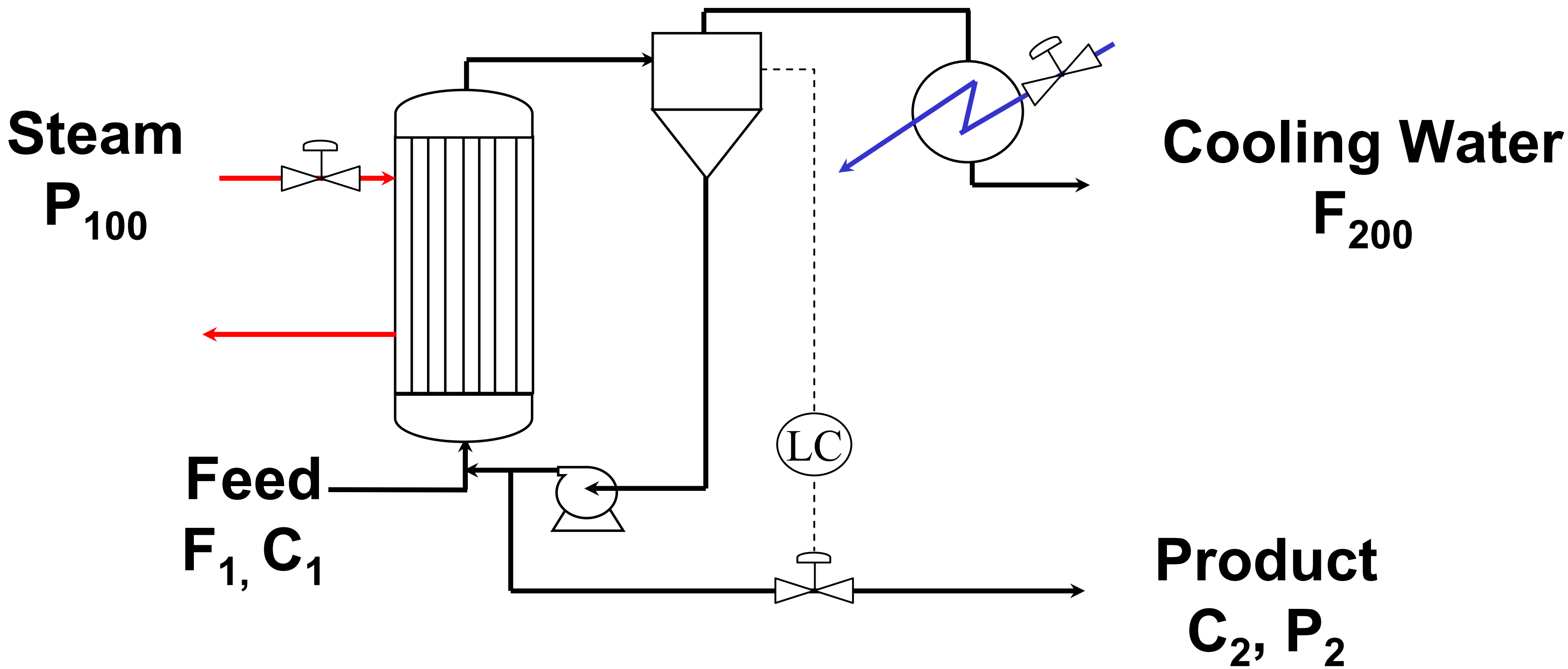
- **The primal and dual solutions are linear functions of θ**
- **The value function is continuous, convex, and quadratic**
- **Developed algorithm for obtaining $z(\theta)$ in closed-form**

MODEL PREDICTIVE CONTROL



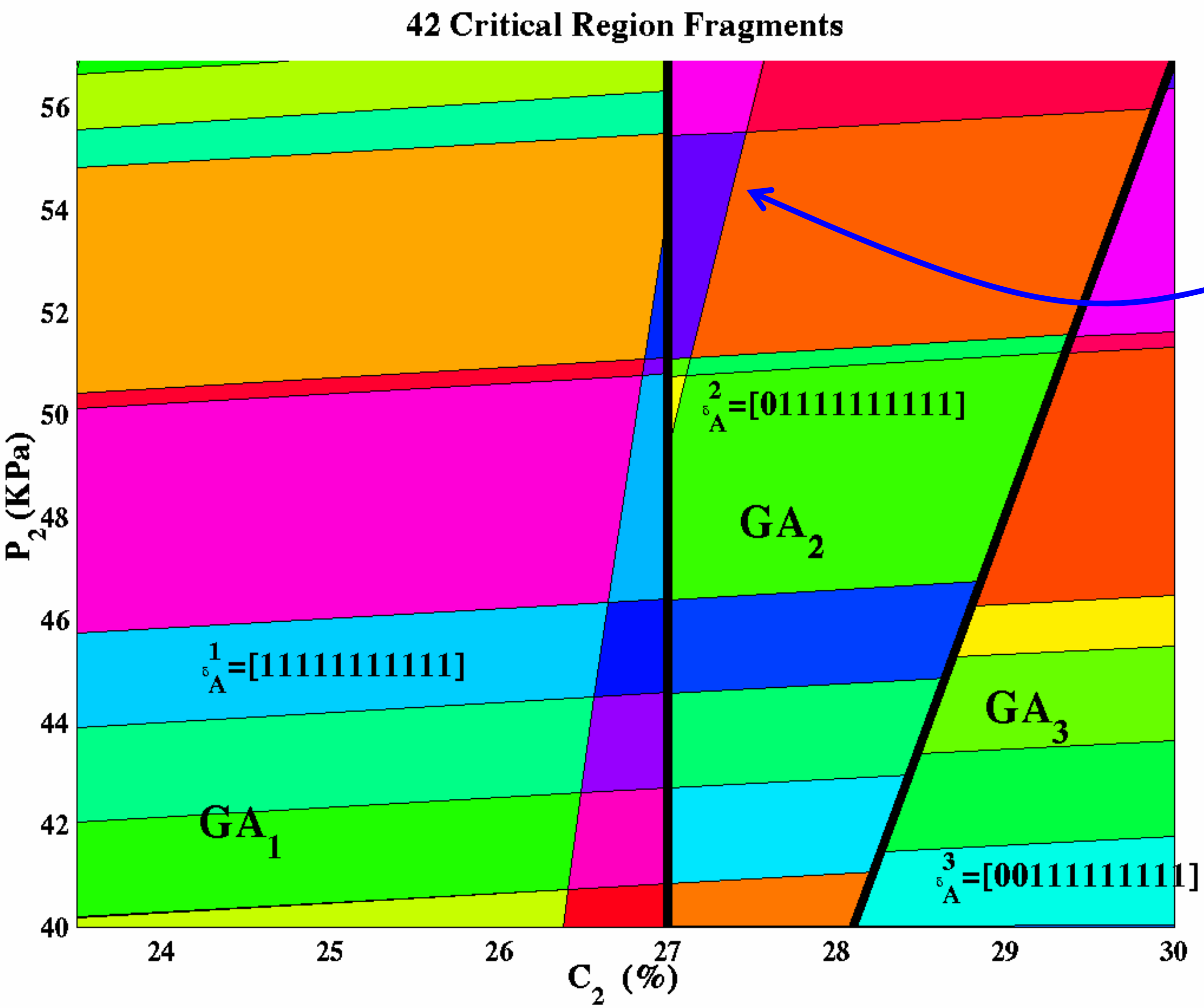
- **Solves an optimization problem at each time interval**

EVAPORATION PROCESS
(Pistikopoulos et al., 2002)



- Control variables: P_{100}, F_{200}
- State variables: C_2, P_2
- Hybrid system

PARAMETRIC SOLUTION



Region

$-0.2\ C_2 \leq -5.4$

$0.0946 P_2 \leq 5.38$

$1.2 C_2 - 0.946 P_2 \leq 27.82$

$0.2 C_2 - 1.16 P_2 \leq -58.89$

Control Law

$$P_{100} = -34.58 C_2 + 2.72 P_2 + 920$$

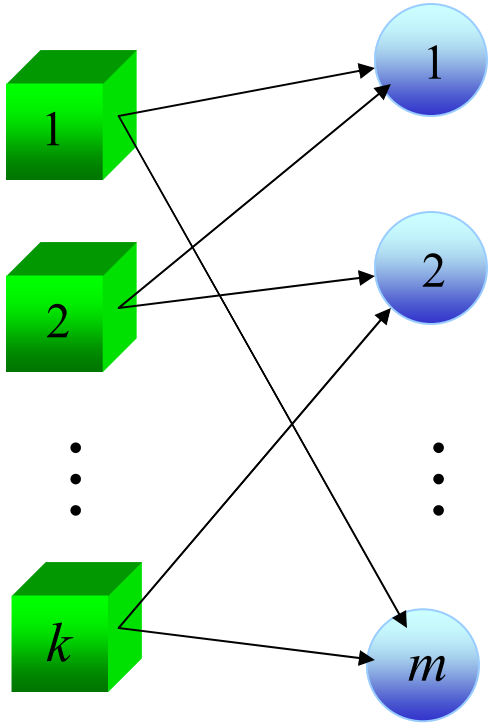
$$F_{200} = 275$$

- ***Explicit*** **control law obtained** ***off-line***
- **You solve only one optimization problem**

PLANNING IN THE SUPPLY CHAIN
Ahmed and Sahinidis (2003)

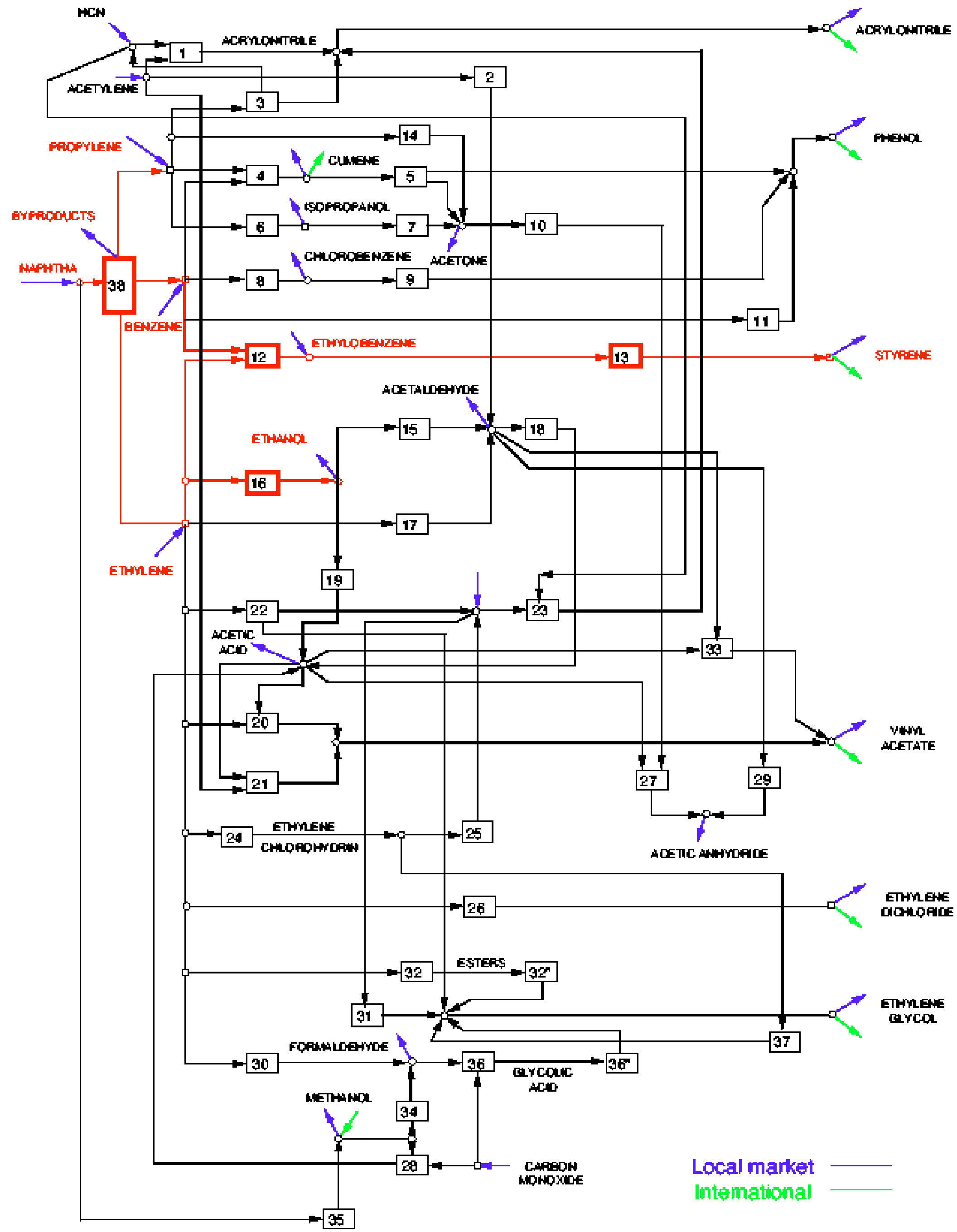
- **Given:**
  - **A network of *k* facilities**
  - ***m* product families**
  - **Forecasts of demands and costs for *n* time periods**

- **Determine**
  - **When and how much to expand?**
  - **How to allocate capacity?**



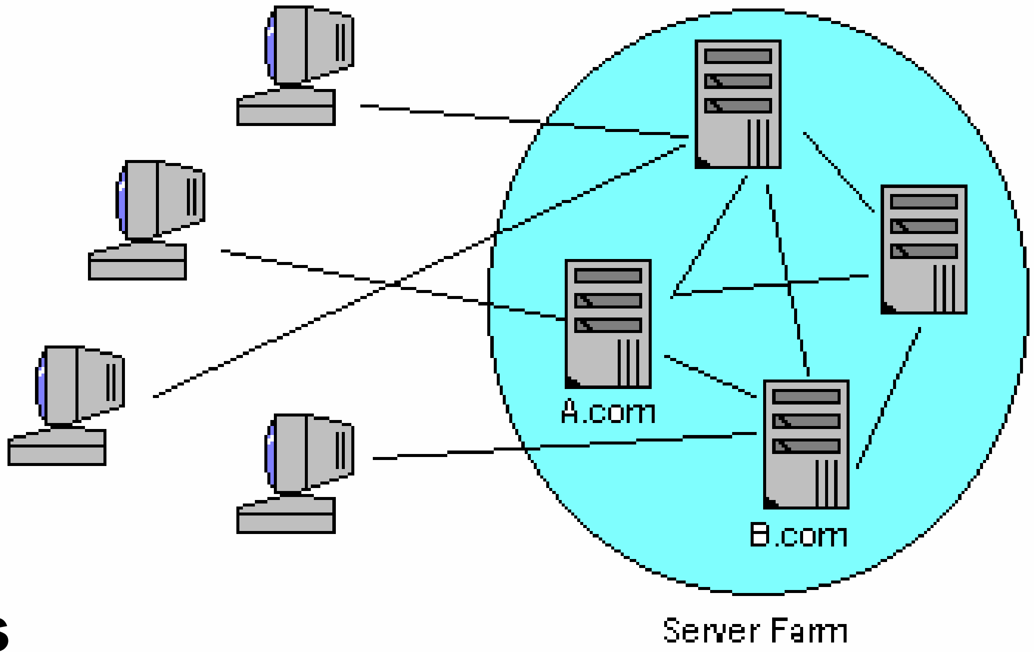
PROCESS SUPPLY CHAIN

- A network of processes, chemicals and markets
- New products and processing technology are anticipated
- When and how much new capacity to bring on-line?



SERVER FARMS

- **A network of servers hosting WebPages**
- **When and how much new technology to install to meet demand growth?**
- **Multi-billion $ industry**
- **Technology adoption is a leading strategic concern**



ASSUMPTIONS

- **Expansion involves a set-up cost $\Rightarrow$ Fixed charge cost function**
- **Linear production technology**
- **No inventories across time periods (can be relaxed)**
- **Continuous expansion units**

THE DETERMINISTIC MILP

$$\min z_n = \sum_{t=1}^{n} \Big[\underbrace{\alpha_t X_t + \beta_t Y_t}_{\text{Expansion Costs}} + \underbrace{tr(\delta_t W_t)}_{\text{Allocation Costs}} \Big]$$

Expansion Costs **Allocation Costs**

Expansion ≤ Bound:	$X_t \leq U_t Y_t$	$t = 1, \ldots, n$
Production ≤ Capacity:	$W_t e \leq X_0 + \sum_{\tau=1}^{t} X_\tau$	
Production = Demand:	$diag(AW_t) = d_t$	
Non-negativity:	$X_t \in \Re_+^k, \; W_t \in \Re_+^{k \times m}$	
Binary Variables:	$Y_t \in \{0,1\}^k$	

UNCERTAINTY

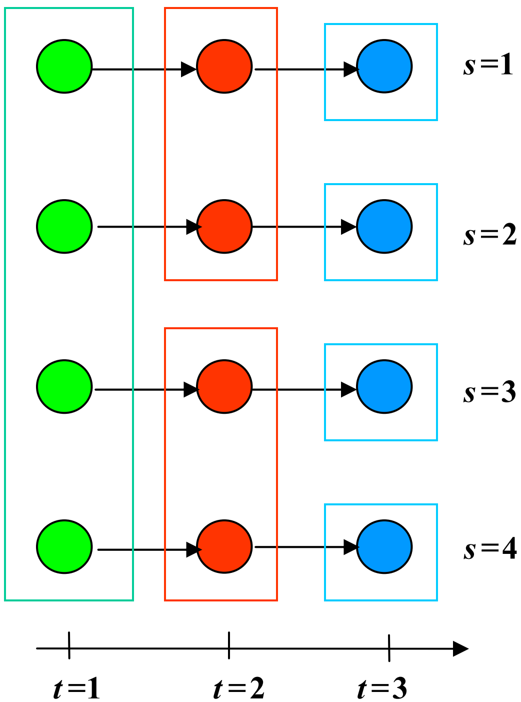
- **Significant forecast uncertainty**
- **Sources:**
  - **Demands**
  - **Costs and prices**
  - **Technology**
- **Evolves over multiple time periods**
- **There are integer decision making variables in every time period/stage**

THE SCENARIO FORMULATION

$$\min \sum_{s=1}^{S} p^s f^s(x^s)$$

$$\text{s.t. } x^s \in \mathsf{X}^s \bigcap \mathsf{N}$$

$$\forall s = 1,\ldots,S.$$



Where

$x^s = (x_1^s, x_2^s, \ldots, x_n^s)$

$x_t^s = (X_t^s, Y_t^s, W_t^s)$

$p^s =$ **Probability of scenario** s

$f^s(\cdot) =$ **Objective function for scenario** s

$\mathsf{X}^s =$ **Constraints for scenario** s

$\mathsf{N} = \{x^s \mid x_t^{s_1} = x_t^{s_2} \;\forall (s_1, s_2) \in \mathsf{B}_t\}$ ← **Non-anticipativity**

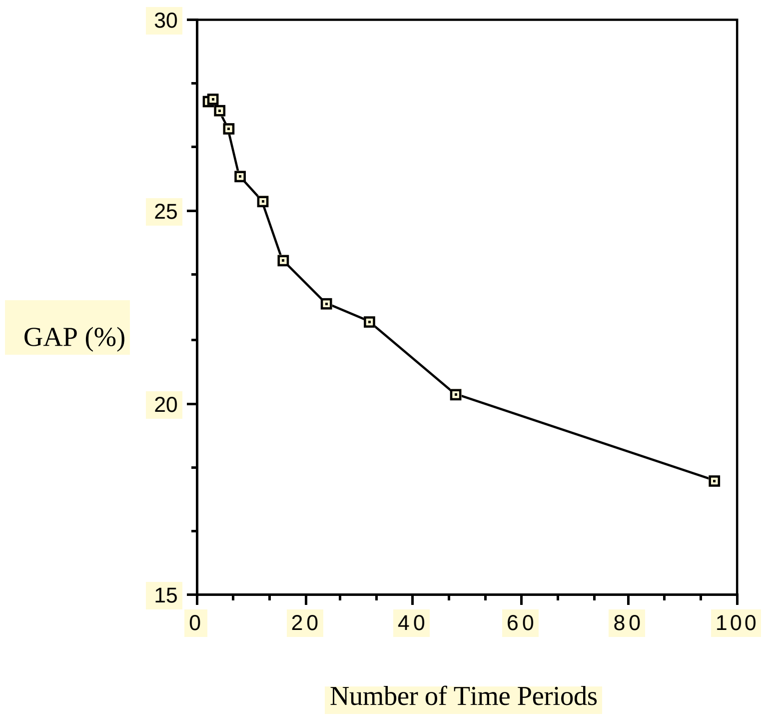
COMPLEXITY IN THE TIME DOMAIN

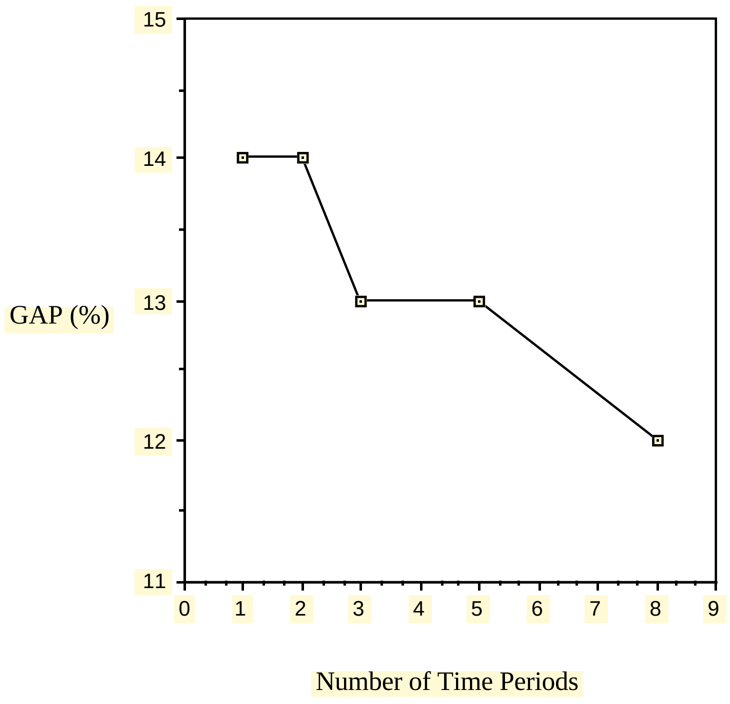
- **The capacitated lot sizing problem (CLSP) is NP-hard**
- **Given any CLSP, we can construct an equivalent instance of the deterministic capacity expansion problem:**

$$\begin{aligned} \min & \sum_{t=1}^{n} (p_t x_t + s_t y_t) \\ s.t. \quad & x_t \le C_t y_t \\ & I_{t-1} + x_t = d_t + I_t \\ & x_t, I_t \ge 0; \; y_t \in \{0,1\} \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \min & \sum_{t=1}^{n} (p_t X_t + s_t Y_t) \\ s.t. \quad & X_t \le C_t Y_t \\ & W_t = \sum_{\tau=1}^{t} d_\tau, \; W_t \le I_0 + \sum_{\tau=1}^{t} X_\tau \\ & X_t, W_t \ge 0; \; Y_t \in \{0,1\} \end{aligned}$$

The deterministic capacity expansion is NP-hard in the number of time periods but…

EMPIRICAL EVIDENCE



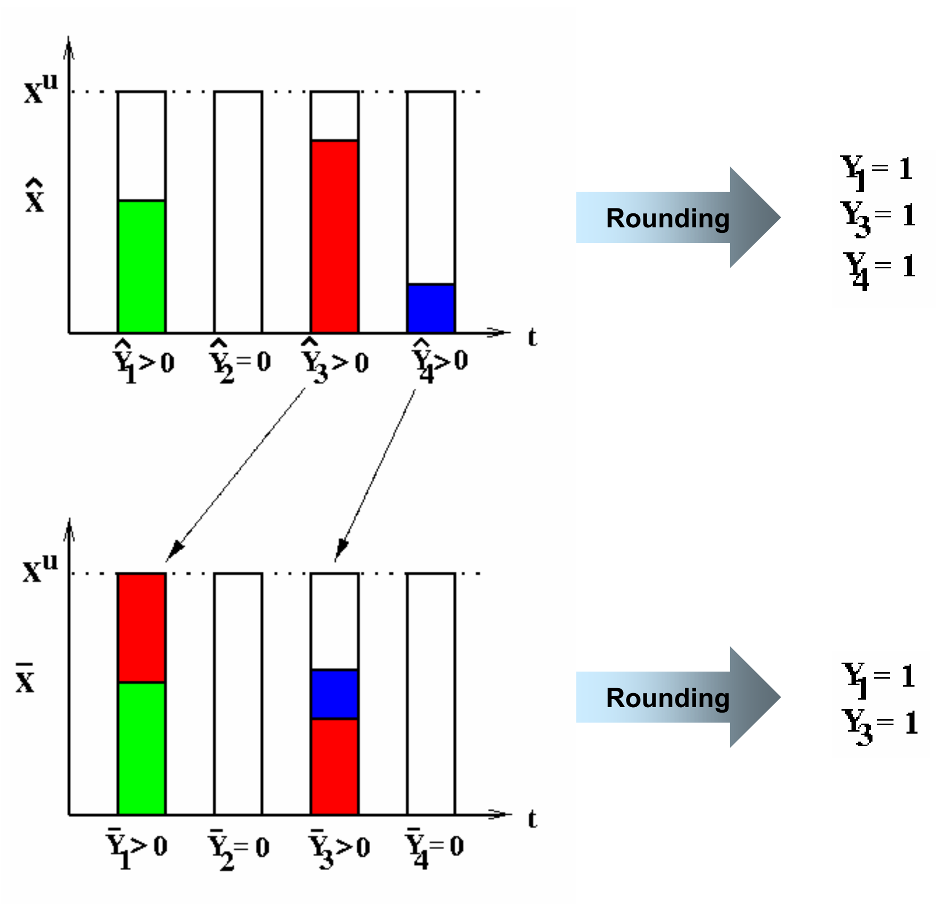


- **Liu & Sahinidis (IECR 1995)**
- **Processing Networks**
- **LP Relaxation**

- **Chang & Gavish (OR 1995)**
- **Telecommunication networks**
- **Lagrangian Relaxation**

CAPACITY SHIFTING

For the Deterministic Problem



N.B.: Naive rounding of LP solution results in too many expansions

3-PHASE HEURISTIC

For the Stochastic Problem

Construct an Implementable solution

- Relax integrality
- Solve as a multi-stage stochastic LP

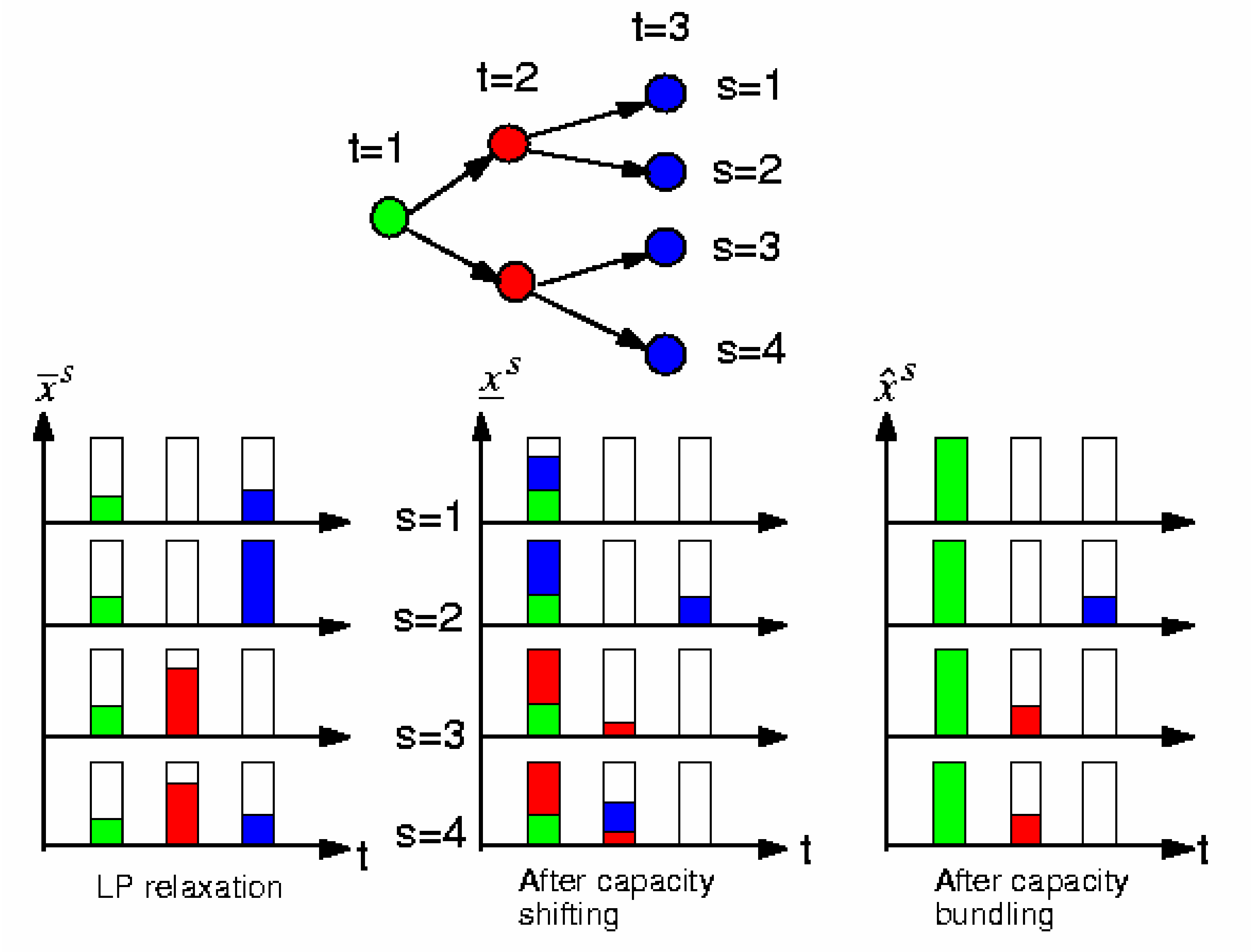
Construct an Admissible solution

- Relax non-anticipativity
- For each scenario, construct an integral solution by capacity shifting

Construct a Feasible solution

- Re-enforce non-anticipativity by capacity bundling

ILLUSTRATION



PROBABILISTIC ANALYSIS

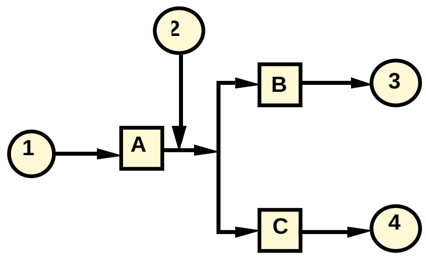
- **How does the heuristic perform in "most" cases?**
- **Consider instances generated from the following probability model:**
  - **Demand in each period is independent with bounded first and second moments**
  - **Cost parameters have bounded distributions**

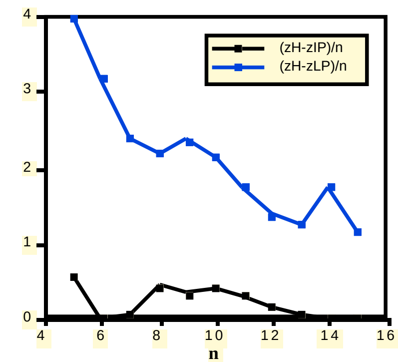
Theorem:

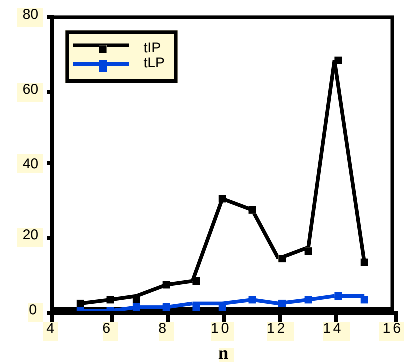
$$\lim_{n\to\infty} \frac{z_n^H - z_n^*}{z_n^*} = 0 \qquad \text{w.p.}\,1$$

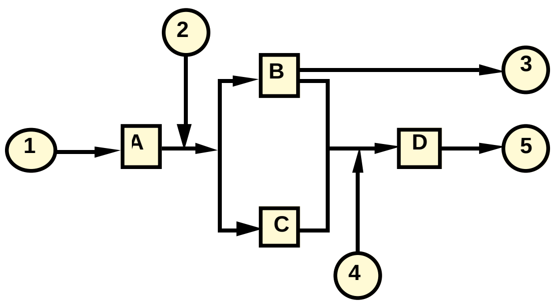
- **For "almost all," "large" sampled instances, the heuristic error vanishes asymptotically**

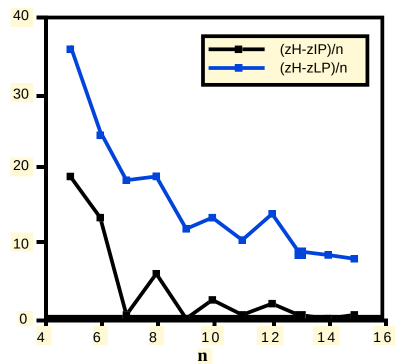
GAPS FOR EXAMPLES 1-3

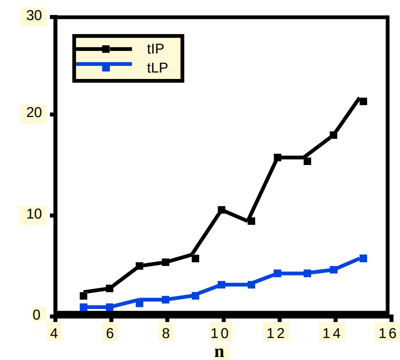


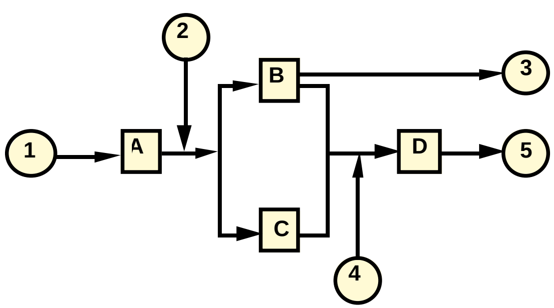


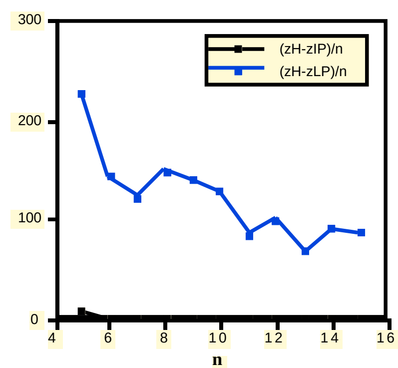


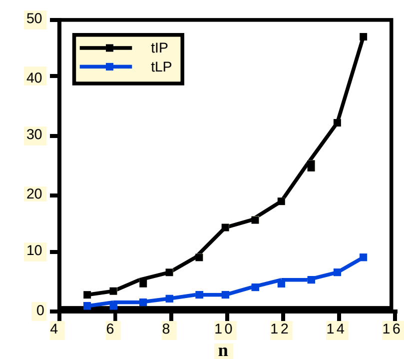




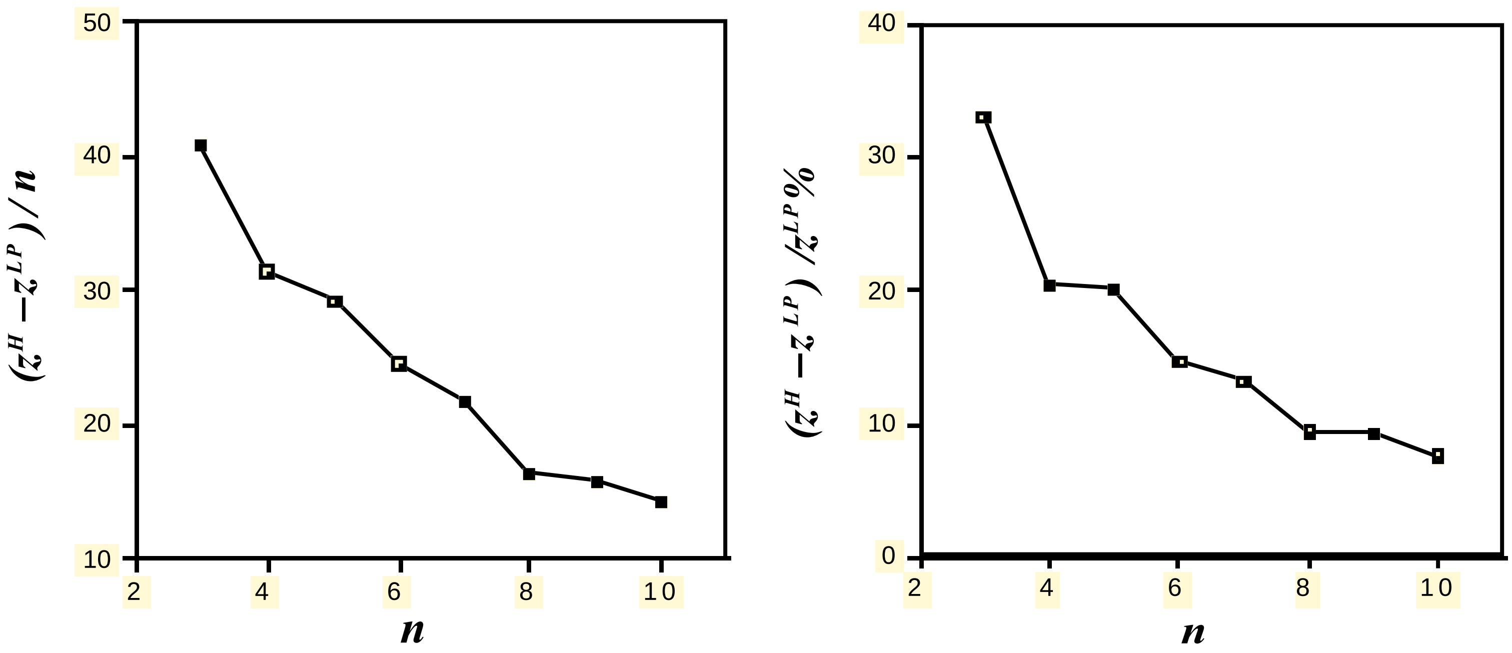








GAPS FOR EXAMPLE 4



- **38 processes, 24 chemicals**
- **With 10 time periods and 2^9 scenarios**
  - **— 36,000 binaries, 184,000 continuous variables, 368,000 constraints**
- **Limited only by the size of the stochastic LP that can be solved**

CONCLUSIONS

"Planning under uncertainty. This, I feel, is the real field we should all be working on."

G. B. Dantzig, E-Optimization (2001)

Optimization algorithm developers *are all* working towards the solution of optimization problems under uncertainty

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